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Supersymmetry and geometric motion

L J Boya†, R F Wehrhahn†‡ and A Rivero†

† Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

‡ Theoretical Nuclear Physics, University of Hamburg, Luruper Chaussee 149, 22761 Hamburg, Federal Republic of Germany

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Abstract. Geometric motion in rank-one symmetric spaces is shown to describe a simple supersymmetric quantum mechanical system. Supersymmetry does indeed lead to a purely algebraic solution for the compact case, providing eigenfunctions and eigenvalues, and also for the Riemannian odd-dimension hyperbolic and Euclidean spaces where SUSY supplies easily the eigenfunctions and hence the phase shifts. In particular, the Jost functions in the latter case are polynomial since the Hamiltonian is seen to be the nth supersymmetric partner of the Hamiltonian of free motion. For the other spaces, supersymmetry proves to be very effective in simplifying and illuminating several aspects of the theory, and suggesting further generalizations.

1. Introduction

According to Einstein, forces originate in geometry; he suggested the study of free, i.e. geodesic motion in curved spaces instead of introducing forces or potentials in ordinary, flat spaces. This geometric approach to interactions, first realized in gravitation theory, pervades much of modern theoretical physics. Exploring this point of view one of us has recently defined symmetry scattering as a comparison between quantum motion in a Riemannian symmetric space $X = G/K$ of non-compact type and the motion on its flat tangential space [1]. The S-matrix was computed explicitly from the general theory of geometric analysis for all Riemannian symmetric spaces [2]; the results were expressed in terms of the multiplicity of the roots of the corresponding Lie algebras. The potential of the equivalent one-dimensional problem could be retrieved by the inverse scattering method from the S-matrix for rank-one spaces, obtaining of course the function of the Laplace–Beltrami operator or a very good approximation to it [3].

On the other hand, another one of us has shown the remarkable features of the quantum mechanical (bound) motion in the compact (still rank-one) case [4].

In this paper we combine these results and consider the supersymmetric approach to treat geometric motion on arbitrary rank-one Riemannian symmetric spaces. To begin, we explain in section 2, why the odd orthogonal hyperbolic spaces $\mathbf{H}^{2n+1} = O(2n+1,1)/O(2n+1)$ have polynomial Jost functions and hence a rational S-matrix; the reason is that the corresponding Hamiltonian is the nth supersymmetric partner of the Hamiltonian of free motion which corresponds to the space $\mathbf{H}^3$. Further, in this section, a connection is made with the other *odd-dimensional spaces, spherical,* $\mathbf{S}^{2n+1}$, *and Euclidean,* $\mathbf{E}_{2n+1}$ *which also have polynomial* Jost functions in appropriate variables; the reason being the same as for the hyperbolic case. Section 3 will explain the general supersymmetric setting of the Laplace–Beltrami operator for arbitrary rank-one symmetric spaces. Writing the Laplacian as $\Delta = (d+\delta)^2$, its supersymmetric form is obvious; Witten [5] has drawn profound consequences from this simple fact.

By relating the Laplacians of different spaces by supersymmetry, one goes a long way towards understanding, for example, the ubiquitous appearance of the gamma functions in the S-matrix; we find supersymmetric relations not only between the even spheres $\mathbb{S}^{2n}$ but also between hyperbolic, complex and quaternionic projective spaces; also the 'octonionic' space, a non-compact form of F_4 is shown to be a supersymmetric partner to $Sp(3,1)/Sp(3)\otimes Sp(1)$ after two supersymmetric steps. In section 4 we study the general compact case; the second quantum number $m_{2\alpha}$ is seen to be related to the spheres $\mathbb{S}^n$, $n = 0, 1, 3, 7$ of norm-one numbers of a division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively. The compact and non-compact cases are essentially related by the change $\sin\theta \mapsto \sinh r$ which, although not quite analytic continuation, is a well studied duality [6]. We conclude this section with a brief look at the Euclidean case: there, the dilatation (in fact, conformal) invariance of the equivalent one-body problem is seen to produce constant phase shifts in the scattering problem. Our final section 5 will contain a comparison with the previous work of the Yale group, namely Alhassid *et al* [7]; they showed in particular how the other homogeneous spaces like $O(2,1)/O(1,1)$, the one-sheeted hyperboloid, are an analytic continuation of spheres and therefore also exhibit a manifest supersymmetric formulation; again, for some cases a complete algebraic solution is obtained by means of supersymmetry. We also wish to mention the work of Olshanetski and Perelomov [8] relating 1D many-body problems to symmetric spaces.

We collect here for the reader's convenience some formulae and data which we are going to use repeatedly later.

A Riemannian symmetric space is a Riemannian manifold with (covariant) constant curvature. Riemannian symmetric spaces are about the simplest manifolds to study; they can all be realized as homogeneous spaces $X = G/K$ where G is a simple Lie group, compact or not, or an 'inhomogeneous group' (in the flat case), and K is a compact subgroup; these objects have been classified since the work of Cartan in the 1920s; there are four series and an exceptional case. The standard mathematical reference is Helgason [6].

Here we refer for definiteness to the rank-one symmetric spaces of the non-compact type; these spaces, X, can be coordinated by a length variable r, $0 \leqslant r < \infty$ and some angles. The Cartan–Killing form in G descends to the quotient $X = G/K$ in the form of the Laplacian ([9], p 267)

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} + m_\alpha \coth r \frac{\mathrm{d}}{\mathrm{d}r} + 2m_{2\alpha} \coth 2r \frac{\mathrm{d}}{\mathrm{d}r} + \Omega \qquad (1.1)$$

where Ω depends only on the angular variables. The integers m_α are related to the multiplicity of the roots of the symmetric space G/K. The following table records their value and other useful information.

Table 1. Parameters for the rank-one Riemannian symmetric spaces of compact type. $\mathbb{H}^n(\mathbb{C})$ is the complex hyperbolic space, $\mathbb{H}^n(\mathbb{H})$ is the quaternionic hyperbolic space and $F_4/O(9)$ is the Cayley plane.

Series	Name	Quotient G/K	Dimension	m_α	$m_{2\alpha}$
$\mathcal{B}_n$	$\mathbb{H}^{2n+1}$	$O(2n+1,1)/O(2n+1)$	$2n+1$	$2n$	0
$\mathcal{D}_n$	$\mathbb{H}^{2n}$	$O(2n,1)/O(2n)$	$2n$	$2n-1$	0
$\mathcal{A}_n$	$\mathbb{H}^n(\mathbb{C})$	$U(n,1)/(U(n)\otimes U(1))$	$2n$	$2(n-1)$	1
$\mathcal{C}_n$	$\mathbb{H}^n(\mathbb{H})$	$Sp(n,1)/(Sp(n)\otimes Sp(1))$	$4n$	$4(n-1)$	3
$\mathcal{F}_4$	Cayley plane	$F_4/O(9)$	16	8	7

2. Motion supersymmetric to free motion

Case (i). Supersymmetric quantum mechanics, shorter SUSY QM, as established by Witten [10], deals in the simplest case with two Hermitian anti-commuting operators, C and Q, with known squares:

$$\{C, Q\} = 0 \qquad C^2 = I \qquad Q^2 = H\,. \tag{2.1}$$

This can be realized as a one-dimensional problem

$$Q(r) = \sigma_2 p_r + \sigma_1 W(r) \qquad C = \sigma_3 \tag{2.2}$$

which produces a SUSY QM pair of Hamiltonians

$$H_\pm = p_r^2 + V_\pm(r) = p_r^2 + W(r)^2 \pm W(r)' \tag{2.3}$$

where p_r is the one-dimensional impulse operator and $W(r)' = \mathrm{d}W(r)/\mathrm{d}r$; these Hamiltonians can also be written as

$$H_+ = AA^+ \qquad \text{and} \qquad H_- = A^+A \tag{2.4}$$

with

$$A = \frac{\mathrm{d}}{\mathrm{d}r} + W(r) \qquad \text{and} \qquad A^+ = -\frac{\mathrm{d}}{\mathrm{d}r} + W(r)\,. \tag{2.5}$$

We wish to consider the SUSY QM partner(s) of the free motion $V = 0$; the regular solutions have been considered before [11]. The equations to solve are

$$W^2(r) \pm W'(r) = \text{const}\,. \tag{2.6}$$

There are five different solutions:

$$W(r) = \begin{cases} 1/r \\ \tan r \\ \tanh r \\ \cot r \\ \coth r\,. \end{cases} \tag{2.7}$$

When the constant in equation (2.6) is set to ± 1 or 0, the free (V = constant) case becomes the first partner. The corresponding five families of phase-invariant potentials in the sense of Gendenshtein [12] are up to a constant term depending on l:

$$V_l(r) = \begin{cases} l(l+1)/r^2 & l = 1, 2, \ldots \\ l(l+1)\sec^2 r & l = 1, 2, \ldots \\ -l(l+1)\mathrm{sech}^2 r & l = 1, 2, \ldots \\ l(l+1)\mathrm{cosec}^2 r & l = 1, 2, \ldots \\ l(l+1)\mathrm{cosech}^2 r & l = 1, 2, \ldots\,. \end{cases} \tag{2.8}$$

All these potentials are completely solvable by algebraic methods, essentially by Q-transforming the free-wavefunction solution for the scattering, cases (i), (iii) and (v), or the kernel of the A operator for the bound states, cases (ii), (iii) and (iv). For example, the solution of the third case [11],

$$V_l(r) = -l(l+1)\mathrm{sech}^2 r$$

has l bound states

$$E^l_m = -m^2 \quad m = 1, 2, \ldots, l \qquad \Psi^l_m(r) = P^l_m(\mathrm{sech}\, r)\,. \tag{2.9}$$

Ψ^l_m are the eigenfunctions of the Hamiltonian with the potential V_l and P^l_m is an associated Legendre polynomial. The S operator, as advertised, is rational:

$$S(k) = (-1)^l \prod_{m=1}^{l} \frac{-\mathrm{i}k + m}{\mathrm{i}k + m}\,. \tag{2.10}$$

All the singularity structure comes from the bound state poles and there is *no* reflection due to the 'half-bound' state at threshold $k = 0$—see [11] and [13].

Case (ii). Now, in previous work [1–3], the potential of case (v) appears for the odd hyperboloid

$$H^{2n+1} = O(2n+1, 1)/O(2n+1)\,.$$

This potential is just

$$V_l(r) = l(l+1)\mathrm{cosech}^2 r \qquad n = l + 1$$

where the 'vacuum' corresponds to $\mathbf{H}^3$; so we now understand why in this case the Jost function is polynomial, and the S operator rational, as remarked in [3]. It is to be noted that $\mathbf{H}^1$ can also be considered, of course, like a free space.

For the even hyperboloids H^{2n} SUSY QM is seen to be realized by

$$W_\lambda(r) = \lambda \tanh r \qquad \lambda = l - \tfrac{1}{2}\,. \tag{2.11}$$

As we shall see in section 3, there is no automatic algebraic solution but all cases are related to one of them, say $\mathbf{H}^2$.

The compact and Euclidean odd cases $\mathbf{S}^{2n+1}$ and $\mathbf{E}^{2n+1}$ are also related to the free motion; duality accounts immediately to this for the spheres, which, if realized as $\mathbf{S}^{2n+1} = O(2n+2)/O(2n+1)$, lead to the potential of case (iv)

$$V_l(r) = l(l+1)\mathrm{cosec}^2 r \tag{2.12}$$

whereas the Euclidean case is realized as case (i) before, namely

$$V(r) = \frac{l(l+1)}{r^2} \qquad \text{connected with } \mathbb{R}^{2n+1} \qquad n = l + 1\,. \tag{2.13}$$

Now, this produces the algebraic formulas for the reduced Bessel functions [14]

$$j_l(r) = (-1)^l \sqrt{\frac{2}{\pi}}\, r^l \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^l \frac{\sin r}{r}\,. \tag{2.14}$$

3. Supersymmetry of geometric scattering

In this section we show that the problem of scattering for rank-one non-compact Riemannian symmetric spaces admits in all cases a SUSY QM formulation, and that these problems are related to each other in a precise way, so one only has to solve a few types.

Consider the quadratic Casimir invariant operator C_2. For these spaces it becomes the Laplacian. The radial part is given by ([9], p 267)

$$\Delta_r = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + m_\alpha \coth r \frac{\mathrm{d}}{\mathrm{d}r} + 2m_{2\alpha} \coth 2r \frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + f(r)\frac{\mathrm{d}}{\mathrm{d}r} \tag{3.1}$$

where the multiplicities were given before in table 1; now in the spectral equation $\Delta_r \Psi(r) = \lambda\Psi(r)$ we can get rid of the first derivative term by the substitution $\Psi = F\Phi$ where

$$2\frac{\mathrm{d}F(r)}{\mathrm{d}r} + F(r)f(r) = 0 \qquad \Rightarrow \qquad F(r) = \exp\left(-\int \frac{f(x)}{2}\mathrm{d}x\right). \tag{3.2}$$

The new Hamiltonian acting on Φ is

$$H\Phi = -\Delta_r\Phi = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(\frac{f(r)}{2}\right)^2 + \frac{1}{2}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right)\Phi \tag{3.3}$$

which shows explicit supersymmetry, with superpotential $W(r)$ and 'ground state' $F(r)$ (cf (2.3)) given by

$$\begin{aligned} &W(r, m_\alpha, m_{2\alpha}) = \frac{f(r)}{2} = \frac{m_\alpha \coth r + 2m_{2\alpha}\coth 2r}{2} \\ &AF(r) = 0 \qquad \text{with} \qquad A = \frac{\mathrm{d}}{\mathrm{d}r} + W(r, m_\alpha, m_{2\alpha}). \end{aligned} \tag{3.4}$$

For the Hamiltonian $H = H_+ = -\mathrm{d}^2/\mathrm{d}r^2 + W(r, m_\alpha, m_{2\alpha})^2 + W(r, m_\alpha, m_{2\alpha})'$ we have

$$H_+ = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_+(r, m_\alpha, m_{2\alpha})$$

with

$$V_+(r, m_\alpha, m_{2\alpha}) = \frac{m_\alpha(m_\alpha + 2m_{2\alpha} - 2)}{4\sinh^2 r} + \frac{m_{2\alpha}(m_{2\alpha} - 2)}{\sinh^2 2r} + \left[\frac{m_\alpha}{2} + m_{2\alpha}\right]^2. \tag{3.5}$$

The partner potential $V_-(r, m_\alpha, m_{2\alpha}) = W(r, m_\alpha, m_{2\alpha})^2 - W(r, m_\alpha, m_{2\alpha})'$ corresponding to the symmetric space having the parameters m_α and $m_{2\alpha}$ can be expressed up to a constant using the potential $V_+(r)$ of another symmetric space. We have

$$V_-(r, m_\alpha, m_{2\alpha}) = \begin{cases} V_+(r, m_\alpha + 2) - 2[m_\alpha + 1] & \text{for } m_{2\alpha} = 0 \\ V_+(r, m_\alpha, m_{2\alpha} + 2) - 2\left[\frac{m_\alpha}{2} + m_{2\alpha} + 2\right] & \text{for } m_{2\alpha} \neq 0. \end{cases} \tag{3.6}$$

By shifting the zero-point energy scale we can get rid of the constant term appearing in the SUSY QM potentials. In the sequel we consider only the r-dependent part of such potentials.

Let us determine now the family of the SUSY QM potential corresponding to each type of symmetric spaces. The orthogonal case is the simplest: for $\mathbf{H}^{2n+1}$ we have $m_\alpha = 2n, m_{2\alpha} = 0$ and

$$W(r,n) = n \coth r \qquad n = 1, 2, \ldots . \tag{3.7}$$

This family yields the quasi-free case considered before with rational S operator.

For $\mathbf{H}^{2n}$ we have $m_\alpha = 2n-1, m_{2\alpha} = 0$, yielding

$$W(r,n) = \left[n - \frac{1}{2}\right]\coth r \qquad \text{and} \qquad A = \frac{\mathrm{d}}{\mathrm{d}r} + \left[n - \frac{1}{2}\right]\coth r \qquad n = 1, 2, \ldots . \tag{3.8}$$

The SUSY chain starts from

$$H_+ = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{4\sinh^2 r} + \frac{1}{4} \qquad (n = 1 \quad m_\alpha = 1 \quad m_{2\alpha} = 0) \tag{3.9}$$

which represents an *attractive* potential; there are no bound states, however, although we are in one dimension, because the singularity at the origin forces us to consider only 'odd' states: the 'ground' state is not allowed in the half-line $r > 0$.

Let $\Phi_2(r)$ be the exact solution of (3.9) with $\Phi_2(0) = 0$, i.e.

$$H_+\Phi_2 = AA^+\Phi_2 = E\Phi_2 . \tag{3.10}$$

Now, using the formalism of SUSY QM we see that $A^+\Phi_2$ is an eigenfunction of H_- for the same eigenvalue $E = k^2$:

$$H_-(A^+\Phi_2) = (A^+A)(A^+\Phi_2) = A^+(AA^+\Phi_2) = E\,A^+\Phi_2 . \tag{3.11}$$

Hence, by repeated application of the operator A^+ on the eigenfunctions on $\mathbf{H}^2$ all the wavefunctions on $\mathbf{H}^{2n}$ are found (cf (3.6)).

This implies that the Jost functions associated with the Schrödinger equation corresponding to a potential determined from the symmetric spaces are also related. From (3.11) we have (the Jost function and the scattering operator are defined here following the convention given for instance in Taylor [15])

$$\Phi_{2(n+1)}(r) = A^+\Phi_{2n}(r) = \left(-\frac{\mathrm{d}}{\mathrm{d}r} + \left(n - \frac{1}{2}\right)\coth r\right)\Phi_{2n}(r) . \tag{3.12}$$

This relation yields asymptotically

$$c_{2(n+1)}(k)\,\mathrm{e}^{-\mathrm{i}kr} = \left(-\frac{\mathrm{d}}{\mathrm{d}r} + n - \frac{1}{2}\right) c_{2n}(k)\,\mathrm{e}^{-\mathrm{i}kr} \tag{3.13}$$

where $c_{2n}(k)$ is the Jost function associated with the Schrödinger equation corresponding to a potential determined from $SO(2n,1)/SO(2)$. We then have

$$c_{2(n+1)}(k) = (\mathrm{i}k + n - \tfrac{1}{2})c_{2n}(k) \tag{3.14}$$

which implies

$$c_{2n}(k) = \prod_{l=i}^{n-1}(\mathrm{i}k - \tfrac{1}{2} + l)c_2(k)\,. \tag{3.15}$$

Hence the appearance of the gamma function is natural:

$$c_{2n}(k) = \frac{\Gamma(\mathrm{i}k + n - \frac{1}{2})}{\Gamma(\mathrm{i}k + \frac{1}{2})}\, c_2(k)\,. \tag{3.16}$$

Therefore, with the initial condition

$$c_2(k) = \frac{\Gamma(\mathrm{i}k + \frac{1}{2})}{\Gamma(\mathrm{i}k)} \tag{3.17}$$

and using for the scattering operator $S_{2n}(k)$ associated with the Hamiltonian H_+ the definition [15] $S_{2n}(k) = c_{2n}(-k)/c_{2n}(k)$, we recover the formula ([3], equation (3.36))

$$S_{2n}(k) = \frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)}\frac{\Gamma(-\mathrm{i}k + n - \frac{1}{2})}{\Gamma(\mathrm{i}k + n - \frac{1}{2})}\,. \tag{3.18}$$

For the unitary and symplectic families the lowest members, $n = 1$, are duals of spheres, namely of $\mathbf{P}^1(\mathbb{C}) = \mathbf{S}^2$ and $\mathbf{P}^1(\mathbb{H}) = \mathbf{S}^4$; hence they are related similarly to the even-spheres chain:

$$\begin{array}{c} (m_\alpha, m_{2\alpha}) = (0,1) \longrightarrow (0,3) \longrightarrow (0,5) \longrightarrow \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \boldsymbol{P}^1(\mathbb{C}) \longrightarrow \boldsymbol{P}^1(\mathbb{H}) \longrightarrow \boldsymbol{S}^6 \longrightarrow . \end{array} \tag{3.19}$$

For example, the dual to $\mathrm{P}^1(\mathbb{C})$, i.e. for $\mathbf{H}^1(\mathbb{C})$, the Hamiltonian turns out to be

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}(2r)^2} - \frac{1}{4\sinh^2 2r} \tag{3.20}$$

corresponding to the non-compact version of S^2 with the scale change $r \longrightarrow 2r$.

For the general case, due to relation (equation (3.6))

$$(m_\alpha, m_{2\alpha}) \longrightarrow (m_\alpha, m_{2\alpha} + 2)$$

we obtain a few more relations:

$$\begin{array}{c} (m_\alpha, m_{2\alpha}) = (4(n-1), 1) \longrightarrow (4(n-1), 3) \\ \mathbf{H}^{2n-1}(\mathbb{C}) \longrightarrow \mathbf{H}^n(\mathbb{H}) \end{array} \tag{3.21}$$

and

$$\begin{array}{c} (m_\alpha, m_{2\alpha}) = (8,1) \longrightarrow (8,3) \longrightarrow (8,5) \longrightarrow (8,7) \\ \mathbf{H}^5(\mathbb{C}) \longrightarrow \mathbf{H}^3(\mathbb{H}) \longrightarrow \ ? \ \longrightarrow \text{Cayley plane}\,. \end{array} \tag{3.22}$$

4. The compact spaces

By duality, formulae of the preceding section hold in the compact case with the obvious substitution *hyperbolic functions* $\longrightarrow$ *spherical functions*, and SUSY QM works just as well; in fact, *better*, because now we can make explicit calculations of the spectrum, degeneracy and wavefunctions in all spaces (not only in the odd spheres) by purely algebraic means. The reason is that the spectrum is purely discrete and infinite, and the ground-state function is calculated by a quadrature (cf (3.4)) and its energy is set to zero. The energy of the excited states are computable from the substitution

$$\begin{cases} (m_\alpha, m_{2\alpha}) \longrightarrow (m_\alpha, m_{2\alpha} + 2\nu) & \text{if } m_{2\alpha} \neq 0 \qquad \nu = 1, 2, 3 \ldots \\ m_\alpha \longrightarrow m_\alpha + 2\nu & \text{if } m_{2\alpha} = 0 \qquad \nu = 1, 2, 3 \ldots . \end{cases} \tag{4.1}$$

Let us work the $\mathbf{S}^2$ case in detail: $m_\alpha = 1$, $m_{2\alpha} = 0$. The (super)potentials and ground-state wavefunction are

$$W(\theta) = \frac{\cot\theta}{2} \qquad V(\theta) = \frac{1}{4\sin^2\theta} \qquad \Psi_0(\theta) = F(\theta)\Phi_0(\theta) = \frac{1}{\sqrt{\sin\theta}}\sqrt{\sin\theta} = \text{const}$$

where $AF(\theta) = 0$ and $A^+\Phi_0(\theta) = 0$. Here $A = \mathrm{d}/\mathrm{d}\theta + \frac{1}{2}\cot\theta$. The spectrum reads

$$E_n = \tfrac{1}{4}[(m_\alpha + 2\nu)^2 - m_\alpha^2] = \nu(\nu+1) \qquad \nu = 0, 1, 2 \ldots$$

as is well known (see e.g. [12]).

The first excited state is proportional to

$$\sqrt{\sin\theta}\left[-\frac{\mathrm{d}}{\mathrm{d}\theta} + \frac{3}{2}\cot\theta\right]\sqrt{\sin\theta} = \cos\theta$$

where in the bracket we use the superpotential of the next ladder, i.e. $W = \frac{3}{2}\cot\theta$.

We now give brief indications for an arbitrary compact Riemannian symmetric rank-one space, as the procedure is completely mechanical; for the spheres $\mathbf{S}^n$ the Laplacian

$$\Delta(\mathbf{S}^n) = \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + (n-1)\cot\theta\frac{\mathrm{d}}{\mathrm{d}\theta} + \Omega \tag{4.2}$$

corresponds to the superpotential

$$W(\theta) = \lambda\cot\theta \qquad \lambda = \frac{n-1}{2}. \tag{4.3}$$

The ground state of the transformed Hamiltonian is obtained using

$$A^+\Phi_0(\theta) = 0 \qquad A^+ = -\frac{\mathrm{d}}{\mathrm{d}\theta} + \lambda\cot,\theta \qquad \Phi_0(\theta) = \sin^\lambda\theta \tag{4.4}$$

whereas the true ground state Ψ_0 in $\mathbf{S}^n$ is (of course) constant, due to the factor $F = \mathrm{e}^{-\int\lambda\cot\theta\mathrm{d}\theta} = \sin^{-\lambda}\theta$ (cf (3.2)); the energy spectrum is

$$E_{\nu,n} = (\lambda+\nu)^2 - \lambda^2 = \nu(\nu+n-1) \qquad \nu = 1, 2, 3 \ldots \tag{4.5}$$

Table 2. Energy spectrum of rank-one Riemannian symmetric spaces of compact type.

Space	m_α	$m_{2\alpha}$	$E(n,\nu)$
$\mathbf{P}^n(\mathbb{R})$	$n-1$	0	$(\frac{1}{2}(n-1)+\nu)\nu$
$\mathbf{P}^n(\mathbb{C})$	$2(n-1)$	1	$(\nu+n)\nu$
$\mathbf{P}^n(\mathbb{H})$	$4(n-1)$	3	$(\nu+2n+1)\nu$
$\mathbf{P}^n(\mathbb{O})$	8	7	$(\nu+11)\nu$

as is well known (e.g. [9], p 16). We could very easily compute the wavefunction of the excited states, but shall refrain from doing so.

We now calculate the spectrum for the other cases using the standard Gendenshtein [12] procedure valid for shape-invariant potentials. From (3.5), (3.6) and (4.1) it follows for the general formula:

$$E_{\nu,m_\alpha,m_{2\alpha}} = \tfrac{1}{4}\tfrac{1}{4}[(m_\alpha + 2(m_{2\alpha}+2\nu))^2 - (m_\alpha + 2m_{2\alpha})^2] = \left(\tfrac{1}{2}m_\alpha + m_{2\alpha} + \nu\right)\nu \tag{4.6}$$

where the extra factor $\frac{1}{4}$ is due to the normalization to radius 1 (this excludes $\mathbf{S}^n$). Table 2 collects the general results.

Intermediate between the elliptic, $\mathbf{S}^n$, and the hyperbolic case, $\mathbf{H}^n$, is the Euclidean one, $\mathbb{R}^n$; here, the absence of curvature produces no scattering; the Laplacian in polar coordinates is

$$\Delta(\mathbb{R}^n) = \frac{d^2}{d\theta^2} + \frac{n-1}{r}\frac{d}{dr} + \frac{1}{r^2}\Omega \tag{4.7}$$

and therefore $W = \frac{1}{2}(n-1)/r$ can act as a superpotential; the conventional potential is

$$V(r) = \frac{(n-1)(n-3)}{4r^2}$$

which is purely centrifugal, repulsive except for $n = 2$, zero for $n = 1, 3$ just as in the other cases **S** and **H**, and produces only constant phase shifts on reflection:

$$\delta_n(k) = \frac{(3-n)}{2}\frac{\pi}{2} \qquad n > 1 \tag{4.8}$$

due to scale invariance [14].

5. Final Remarks

A powerful algebraic theory of scattering has been developed by the Yale group [7]; they can also treat some pseudo-Riemannian spaces which were excluded from our treatment. The simplest case they study corresponds to the one-sheeted hyperboloid, $SO(2,1)/SO(1,1)$. The Laplacian

$$\Delta = \frac{d^2}{dr^2} + \frac{\lambda(\lambda+1)}{\cosh^2 r} \tag{5.1}$$

where λ can be integer or half-integer, corresponds to our quasi-free case 2 of section 2 (for λ integer).

The Hamiltonian associated with (5.1) has both discrete and continuous spectra. They represent, respectively, the quantum aspects of closed and open geodesics in the one-sheeted hyperboloid; by contrast, the two-sheeted hyperboloid has a purely scattering potential, and the classical geodesics are all open. For a recent re-examination of the algebraic scattering see [16].

Olshanetski and Perelomov [8] consider one-dimensional classical and quantum systems interacting pairwise via potentials of the five types: (i) $V(r) = 1/r^2$, (ii) $V(r) = a^2/\sinh^2 ar$, (iii) $V(r) = a^2/\sin^2 ar$, (iv) $V(r) = a^2\wp(ar)$, (v) $V(r) = 1/r^2 + \omega^2 r^2$, where $\wp$ is the Weierstrass $\wp$-function. Cases (i)–(iii) correspond to our symmetric spaces; in fact, in [8] these potentials are obtained as projections from a symmetric space. The SUSY QM of the harmonic oscillator is well known; it suffices to take as superpotential $W(r) = r + b/r$. The connection to the Weierstrass function requires additionally techniques of algebraic geometry.

There are several directions in which our work can be extended. An obvious one is higher-rank spaces; another is to study different quotient spaces, as in the Yale group. Several of these lines are now being explored.

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